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REPORT NO. 1077

MAY 1959

JUMP DUE TO AERODYNAMIC ASYMMETRY OF A MISSILE WITH VARYING ROLL RATE

C. H. MURPHY

J. W. BRADLEY

DEPARTMENT OF THE ARMY PROJECT NO. 5U3-03-001

ORDNANCE RESEARCH AND DEVELOPMENT PROJECT NO. TB3-0108

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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ABSTRACT

Although the theory of aerodynamic jump for essentially symmetric missiles with slight aerodynamic asymmetry and constant roll is well known, the dependence of this jump on varying roll has not been considered. If roll is produced by differentially canted controls, the rolling motion can be described by two parameters. The magnitude and orientation of jump is presented for the pertinent range of these parameters.

TABLE OF SYMBOLS

A = $\phi_\infty' \, C^{-1}$, a measure of the angle turned through as the roll rate approaches steady-state

B = ϕ_0' / ϕ_∞'

C is roll damping coefficient

C_D is drag coefficient

C_{ℓ_p} is roll moment coefficient due to roll

C_{ℓ_δ} is roll moment coefficient due to cant

C_{L_α} = $C_{N_\alpha} - C_D$

C_{M_0} is moment coefficient due to aerodynamic asymmetry

C_{M_q} is damping moment coefficient

C_{M_α} is static moment coefficient

$C_{M_{\dot\alpha}}$ is moment coefficient due to cross-acceleration

C_{M_ϵ} is moment coefficient due to control surface deflection

C_{N_0} is normal force coefficient due to aerodynamic asymmetry

C_{N_α} is normal force coefficient

C_{N_ϵ} is normal force coefficient due to control surface deflection

$$E = -\left[\lim_{s \to \infty} \frac{1}{s} \int_0^s \left(J_{\tilde\xi}' \, \tilde\xi' + J_{\tilde\xi} \, \ddot{\tilde\xi}\right) ds_2\right]$$

$$f(r) = \frac{1}{A} \phi \left(iC^{-1} r\right)$$

$$H = \frac{\rho S \ell}{2m} \left[C_{L_\alpha} - C_D - k_t^{-2} \left(C_{M_q} + C_{M_{\dot\alpha}}\right)\right]$$

I_x is axial moment of inertia

$I_y = I_z$ are transverse moments of inertia

$$J_A = \frac{\rho S \ell}{2m} \left[C_{N_0} \, e^{i\phi_N} - \frac{C_{L_\alpha} C_{M_0}}{C_{M_\alpha}} \, e^{i\phi_M}\right]$$

$J_{\xi}^{\sim} \;=\; HJ_{\xi}^{\sim\prime} \;\doteq\; 0$

$J_{\xi}^{\sim\prime} \;=\; k_t^{\,2}\,\dfrac{C_{L_\alpha}}{C_{M_\alpha}}$

k_a is dimensionless axial radius of gyration

k_t is dimensionless transverse radius of gyration

ℓ is reference length

L is roll moment

m is mass

$M \;=\; \dfrac{\rho S \ell}{2m}\, k_t^{-2}\, C_{M_\alpha}$

$M_A \;=\; -\,\dfrac{\rho S \ell}{2m}\, k_t^{-2}\, C_{M_0}$

$\tilde{M} + i\tilde{N}$ is linear aerodynamic moment

$p \;=\; \phi'$, the roll rate

$\tilde{q} + i\tilde{r}$ is transverse angular velocity

$s \;=\; \int \frac{V}{\ell}\, dt$, dimensionless distance along flight path

$\hat{s} \;=\; Cs$

S is reference area

t is time

V is velocity

x, y, z are components of a space-fixed coordinate system defined in the text

$\tilde{Y} + i\tilde{Z}$ is transverse aerodynamic force

$\tilde{\alpha}$ is angle of attack

$\tilde{\beta}$ is angle of sideslip

δ is cant angle

ϵ		is angle of deflection of control surface
$\tilde{\xi}$	=	$\tilde{\beta} + i\tilde{\alpha}$, the complex angle of attack
ρ		is air density
ϕ		is roll angle
ϕ_0'		is initial roll rate
ϕ_∞'		is steady-state roll rate, defined to be positive
ϕ_M		is initial orientation angle of asymmetric moment
ϕ_N		is initial orientation angle of asymmetric force
ϕ_ϵ		is initial orientation angle for asymmetry due to ϵ
Φ	=	$\lim_{s\to\infty}\int_0^s \int_0^{s_2} e^{i\phi}\, ds_1\, ds_2$, the coefficient of J_A in the jump equation
$\hat{\Phi}$	=	$\phi_\infty'\Phi = iA\int_0^\infty e^{Af(r)}\, dr$, is the ratio of Φ to the magnitude of constant-spin Φ
$(\ )'$		denotes derivative with respect to s
$(\tilde{\ })$		indicates forces and moments are measured in non-rolling coordinate system

INTRODUCTION

The jump angle is defined to be the angle between the launch direction of a missile and its "effective" line of departure. (The effective line of departure is the line joining the launch point and a distant point on a gravity-free trajectory.) In Reference 1 relations for the jump due to aerodynamic forces are derived for an essentially symmetric missile with a slight configurational asymmetry and constant roll rate. Unfortunately most missiles of this type build up their roll over an initial portion of their flight path and so this analysis is not directly applicable. In this report we will consider the influence of a varying roll rate on aerodynamic jump.

ROLL EQUATION

It will be assumed that the only aerodynamic moments associated with the rolling motion are a roll inducing moment caused by a differential cant and a roll damping moment.

$$\therefore L = \left(\tfrac{1}{2}\right) \rho V^2 S \ell \left[C_{\ell_p} \left(\frac{p\ell}{V}\right) + C_{\ell_\delta} \delta \right] \qquad (1)$$

where L is roll moment

ρ is air density

V is velocity

S is reference area

ℓ is reference length*

C_{ℓ_p} is roll moment coefficient due to roll

C_{ℓ_δ} is roll moment coefficient due to cant

p is roll rate and

δ is cant angle

---

*ℓ as a subscript refers to the roll moment.

From this roll moment the following equation for the roll angle can be derived.[2] (The roll angle at $s = 0$ was selected to be zero.)

$$\phi(s) = \phi_\infty' s + \left(\frac{\phi_\infty' - \phi_0'}{C}\right)\left(e^{-Cs} - 1\right) \qquad (2)$$

where ϕ is roll angle

$\phi_0' = \left(\frac{d\phi}{ds}\right)_0$ is initial roll rate

$s = \int \frac{V}{\ell}\, dt$ is dimensionless distance along flight path

$\phi_\infty' = \dfrac{-C_{\ell_\delta}\,\delta}{C_{\ell_p} + k_a^2\, C_D}$ is steady state roll rate

$k_a = \sqrt{\dfrac{I_x}{m\ell^2}}$ is dimensionless axial radius of gyration

C_D is drag coefficient and

$C = -\left(\frac{\rho S \ell}{2m}\right)\left(k_a^{-2}\, C_{\ell_p} + C_D\right)$ is roll damping coefficient

In order to avoid consideration of both positive and negative values of the steady state roll rate, we will select our space-fixed coordinate system in a rather special way. The positive direction of the x-axis is defined as the launch direction. If the roll angle is measured from the positive y-axis, then ϕ_∞' is positive or negative accordingly as steady state roll has the same or opposite direction, respectively, of a 90° rotation from positive y-axis to positive z-axis. Thus we can restrict ϕ_∞' to positive values* without loss of generality by defining the positive z-axis as that axis obtained by a 90° rotation of the fixed positive y-axis in the direction of steady state roll. This celection of coordinate axes will, therefore, be used throughout this report.

Since we intend to study the effect of different rolling motions on the aerodynamic jump, it would be desirable to reduce the number of parameters in Equation (2). This set of three parameters can be reduced

*The special case of zero steady-state roll corresponds to a parabolic deflection of the missile instead of the linear deflection described by jump. We will not consider the case $\phi_\infty' = 0$ in this report.

to a set of two by properly selecting the scale for s. This can be done by the transformation $s = C^{-1}\hat{s}$.

$$\phi(C^{-1}\hat{s}) = A\left[\hat{s} - (B-1)(e^{-\hat{s}} - 1)\right] \tag{3}$$

where $A = \phi_{\infty}' C^{-1}$

$B = \phi_0'/\phi_{\infty}'$

As can be seen from Equation (2), C^{-1} is the distance required for the roll rate to reach a value $.37|\phi_{\infty}' - \phi_0'|$ units from its steady state value. A, then, is a measure* of the angle turned through as the roll rate approaches ϕ_{∞}'. In the special case $A = 0$, which implies $C^{-1} = 0$, the transformation to Equation (3) is invalid. Since Equation (2) reduces to $\phi(s) = \phi_{\infty}'s$ for either $C^{-1} = 0$ or $\phi_{\infty}' = \phi_0'$, however, we can consider the case $A = 0$ as equivalent to $A \neq 0$, $B = 1$.

AERODYNAMIC JUMP

The major linear aerodynamic forces which cause aerodynamic jump are defined by the following equations [1,3]:

$$\text{Drag} = (1/2)\,\rho V^2 S\, C_D \tag{4}$$

$$\tilde{Y} + i\tilde{Z} = (1/2)\,\rho V^2 S\left[-C_{N_\alpha}\tilde{\xi} + C_{N_0}\, e^{i(\phi + \phi_N)}\right] \tag{5}$$

where $\tilde{Y}$ and $\tilde{Z}$ are transverse components of aerodynamic force

$\tilde{\xi} = \tilde{\beta} + i\,\tilde{\alpha}$ is complex angle of attack

$\sim$ indicates forces and moments are measured in non-rolling coordinate system

C_{N_0} is dimensionless normal force coefficient due to aerodynamic asymmetry

ϕ_N is initial orientation angle of asymmetric force

If the missile axis always makes a small angle with the launch direction, the following simple differential equation[1] controls the lateral motion induced by aerodynamic forces:

*Since ϕ_{∞}' is measured in radians per distance and C^{-1} is measured in units of distance, A is given in terms of radian measure.

$$\frac{y'' + i z''}{\ell} = \frac{\rho S \ell}{2m}\left[- C_{L_\alpha} \tilde{\xi} + C_{N_0} e^{i(\phi + \phi_N)}\right] \tag{6}$$

where $C_{L_\alpha} = C_{N_\alpha} - C_D$

and primes denote derivatives with respect to s.

The aerodynamic jump can be defined analytically by the equation

$$\text{Aero. jump} = \lim_{x \to \infty} \frac{y + i z}{x} = \lim_{s \to \infty} \frac{y + i z}{\ell s}$$

$$= \frac{\rho S \ell}{2m} \lim_{s \to \infty} \frac{1}{s} \int_0^s \int_0^{s_2} \left[-C_{L_\alpha} \tilde{\xi} + C_{N_0} e^{i(\phi + \phi_N)}\right] ds_1\, ds_2 \tag{7}$$

Thus the aerodynamic jump depends on $\tilde{\xi}$ and ϕ. ϕ is given by Equation (2) and $\tilde{\xi}$ can be computed from the aerodynamic moment. Since jump due to aerodynamic asymmetry is small for high roll rates, low roll rates will be assumed and the Magnus moment thereby neglected. The linear aerodynamic moment is defined by the equation:

$$\tilde{M} + i \tilde{N} = (1/2)\rho V^2 S \ell \left[- i C_{M_\alpha} \tilde{\xi} + C_{M_q} \frac{(\tilde{q} + i \tilde{r})\ell}{V} - i C_{M_{\dot\alpha}} \tilde{\xi}' + i C_{M_0} e^{i(\phi + \phi_M)}\right] \tag{8}$$

where $\tilde{q}$, $\tilde{r}$ are transverse components of angular velocity

C_{M_0} is dimensionless moment coefficient due to aerodynamic asymmetry

ϕ_M is initial orientation angle of asymmetric moment

Equations (4, 5, 8) can be placed in the dynamic equations and the usual equation of pitching and yawing motion derived[3,4]. (Due to the assumption of small roll rate, the gyroscopic terms drop out.)

$$\tilde{\xi}'' + H \tilde{\xi}' - M \tilde{\xi} = M_A e^{i(\phi + \phi_M)} \tag{9}$$

$$\text{where } H = \frac{\rho S \ell}{2m}\left[C_{L_\alpha} - C_D - k_t^{-2}\left(C_{M_q} + C_{M_{\dot\alpha}}\right)\right]$$

$$M = \frac{\rho S \ell}{2m} k_t^{-2} C_{M_\alpha}$$

$$M_A = -\frac{\rho S \ell}{2m} k_t^{-2} C_{M_0}$$

$$k_t = \sqrt{\frac{I_y}{m\ell^2}} = \sqrt{\frac{I_z}{m\ell^2}}$$ is dimensionless transverse radius of gyration

Equation (9) can be solved for $\tilde{\xi}$.

$$\tilde{\xi} = M^{-1}\left[\tilde{\xi}'' + H\tilde{\xi}' - M_A e^{i(\phi + \phi_M)}\right] \tag{10}$$

$\tilde{\xi}$ from Equation (10) can be substituted into Equation (7) and the result can then be simplified to

$$\text{Aero. Jump} = J_{\tilde{\xi}'}\tilde{\xi}_0' + J_{\tilde{\xi}}\tilde{\xi}_0 + J_A \Phi + E \tag{11}$$

$$\text{where } J_{\tilde{\xi}'} = k_t^2 \frac{C_{L_\alpha}}{C_{M_\alpha}}$$

$$J_{\tilde{\xi}} = H J_{\tilde{\xi}'} \doteq 0$$

$$J_A = \frac{\rho S \ell}{2m}\left[C_{M_0} e^{i\phi_M} - \frac{C_{L_\alpha} C_{M_0}}{C_{M_\alpha}} e^{i\phi_M}\right]$$

$$\Phi = \lim_{s\to\infty} \frac{1}{s}\int_0^s \int_0^{s_2} e^{i\phi}\, ds_1\, ds_2$$

$$E = -\left[\lim_{s\to\infty}\frac{1}{s}\int_0^s \left(J_{\tilde{\xi}'}\tilde{\xi}' + J_{\tilde{\xi}}\tilde{\xi}\right) ds_2\right]$$

If the missile is dynamically stable, $\tilde{\xi}$ and its integral are bounded.
$\therefore E = 0$.

For a statically stable missile $J_{\tilde{\xi}}{}'$ is negative. In conformity with aerodynamic conventions[5], however, the missile's nose points along the negative y-axis for positive β. (This can be seen from Equation (5) which states that the direction of the complex normal force for the usual symmetric missile ($C_{N_\alpha} > 0$, $C_{N_0} = 0$), is the negative of that of the complex angle of attack.) Thus the jump due to $\tilde{\xi}_0{}'$ is in the direction of $\tilde{\xi}_0{}'$.

If the aerodynamic asymmetry arises from a single control surface deflected at an angle ϵ with initial orientation angle ϕ_ϵ, J_A reduces to a simple form:

$$J_A = \frac{\rho S \ell}{2m}\left[C_{N_\epsilon} - \frac{C_{L_\alpha} C_{M_\epsilon}}{C_{M_\alpha}}\right]\epsilon\, e^{i\phi_\epsilon} \qquad (12)$$

Equation (12) reveals the conclusion stated in Reference 1 that the jump due to deflected control surface is zero when the control surface is located near the center of pressure of the complete configuration, i.e. $\left(\frac{C_{M_\alpha}}{C_{L_\alpha}}\right)\ell$ units from the center of mass. Figure 1 plots y/x versus z/x for representative rolling motions, with $\tilde{\xi}_0{}' = \tilde{\xi}_0 = E = 0$, where

$$\text{Aero. Jump} = \lim_{x\to\infty} \frac{y + i z}{x} = J_A \Phi$$

and Φ has the magnitude and orientation of the limit points. The remainder of this report will concern itself with Φ, the coefficient of J_A.

SIMPLIFICATION OF THE FUNCTION Φ

In order to consider the influence of varying roll on Φ, we will simplify it by the following algebraic steps:

$$\Phi = \lim_{s\to\infty} \frac{1}{s}\int_0^s \int_0^{s_2} e^{i\phi(s_1)}\, ds_1\, ds_2$$

$$= \lim_{s\to\infty} \frac{1}{s}\int_0^s \int_{s_1}^{s} e^{i\phi(s_1)}\, ds_2\, ds_1$$

$$= \lim_{s\to\infty} \int_0^s \left(1 - \frac{s_1}{s}\right) e^{i\phi(s_1)}\, ds_1 \qquad (13)$$

By use of a contour integration in the complex plane and the transformation of the roll equation used for Equation (3), Equation (13) can be considerably simplified. The mathematical details are given in Appendix A.

$$\therefore \Phi = \frac{iA}{\phi_{\infty}'} \int_0^{\infty} e^{Af(r)}\, dr \qquad (14)$$

$$\text{where} \quad f(r) = \frac{i}{A}\phi\,(iC^{-1}r) = -\left[r + i\,(B-1)\,(e^{-ir}-1)\right]$$

For the constant rolling motion considered in Reference 1, B is unity and

$$\Phi = i\,(\phi_{\infty}')^{-1} \qquad (15)$$

Thus for a single deflected control surface the jump has a magnitude $\frac{\rho S \ell}{2m}\left| C_{N_\epsilon} - \frac{C_{M_\epsilon} C_{L_\alpha}}{C_{M_\alpha}} \right| \epsilon \; (\phi_{\infty}')^{-1}$ and orientation at a right angle to the initial orientation of the control surface. This orientation is in the direction of the spin if $C_{N_\epsilon} - \frac{C_{M_\epsilon} C_{L_\alpha}}{C_{M_\alpha}}$ is positive and opposite if this quantity is negative.

We will use the magnitude of this jump for constant spin as our standard and compare the magnitudes of the jumps for varying spin with it. For this reason the following definition is introduced.

$$\begin{aligned} \hat{\Phi} &= \phi_{\infty}' \Phi \\ &= iA \int_0^{\infty} e^{Af(r)}\, dr \\ &= \hat{\Phi}(A, B) \end{aligned} \qquad (16)$$

Direct numerical calculations of $\hat{\Phi}$ are possible from Equation (16) by both a power series expansion (Appendix B) and an asymptotic series expansion (Appendix C). The former converges rapidly when A is small, the

latter when A is large. Before proceeding to this calculation, it is instructive to see what can be predicted concerning the behavior of $\hat{\Phi}$.

PREDICTED BEHAVIOR OF $\hat{\Phi}$

For constant roll, by definition

$$\hat{\Phi}(A, 1) = 1 \tag{17}$$

By the simple analysis following Equation (3), $\hat{\Phi}(0,B)$ must equal $\hat{\Phi}(A, 1)$. Since $\hat{\Phi}$ is a continuous function of A, we have some indication of the behaviour of $\hat{\Phi}$ in the vicinity of zero A:

$$\lim_{A \to 0} \hat{\Phi}(A,B) = 1 \tag{18}$$

The larger part of aerodynamic jump occurs during the first few revolutions. (Figure 1 illustrates this fact.) From the physical interpretation of A, it can be seen that when A is large the spin nears steady state slowly, so that over the first few revolutions, the spin can be regarded as a constant, equal to its initial value. Thus, replacing ϕ_∞' by ϕ_0' in Equation (15),

$$\hat{\Phi}(A,B) \doteq \frac{1}{B}, \quad B \gg 1 \tag{19}$$

For a fixed A, the smaller the value of ϕ_0', the greater will be the change in spin during the first few revolutions. Thus we would expect this approximation to become poorer as $B \to 0$. (This rather intuitive reasoning is substantiated by the more rigorous mathematical analysis in Appendix C.)

A special case of some interest is that of zero initial roll. For this case

$$\phi\,(C^{-1}\hat{s}) = A\left(\frac{\hat{s}^2}{2} - \frac{\hat{s}^3}{6} + \ldots\right) \tag{20}$$

$$\therefore\; f(r) = i\left(-\frac{r^2}{2} + \frac{ir^3}{6} + \ldots\right) \tag{21}$$

If A is sufficiently large, ϕ will reach several revolutions before the cubic term in Equation (20) has an important effect. Therefore, for large A,

ϕ can be approximated by the quadratic term over the regime where jump occurs. Under this assumption, we have

$$\hat{\Phi}(A,0) \doteq iA\int_0^\infty e^{-\frac{iAr^2}{2}}\,dr$$

$$= i\sqrt{2A}\left[\int_0^\infty \left(\cos\frac{Ar^2}{2} - i\sin\frac{Ar^2}{2}\right)\sqrt{\frac{A}{2}}\,dr\right] \qquad (22)$$

Using the complete Fresnel integral:

$$\hat{\Phi}(A,0) = i\sqrt{2A}\left[\frac{1}{2}\sqrt{\frac{\pi}{2}}(1-i)\right]$$

$$= \frac{1}{2}\sqrt{\pi A}\;(1+i)$$

$$= \sqrt{\frac{\pi A}{2}}\;e^{i\frac{\pi}{4}}$$

$$= 1.25\sqrt{A}\;e^{i(45^\circ)} \qquad (23)$$

Thus this approximation* predicts that the jump due to aerodynamic asymmetry makes a 45° angle with the initial orientation when the initial spin rate is zero.

COMPUTED BEHAVIOR OF $\hat{\Phi}$

Numerical solutions of the expansions in Appendices B and C were obtained on the ORDVAC for a variety of values of A and B. The results are plotted in Figures 2, 3 and 4. Figures 2 and 3 show the magnitude and orientation, respectively, of $\hat{\Phi}$ as a function of A for various values of B. For small values of the parameter A, the behavior of $\hat{\Phi}$ is shown more clearly by Figure 4 which plots both the magnitude and orientation of $\hat{\Phi}$ in polar coordinates. From a consideration of these figures, we can determine the reliability of the approximations expressed by Equations (19) and (23).

---

*Equation (23) has been obtained in a different manner jointly by C. L. Poor and B. G. Karpov.

Equation (19) applies for values of A as low as π when $B > 1$. For $B < 1$, the lower bound of the usable range of A grows rapidly with decreasing B.

The approximation of Equation (23) is reasonably accurate for $A > \frac{\pi}{4}$ as regards magnitude and for $A > 2\pi$ as regards orientation. However, the nearly constant difference between the computed and predicted magnitudes, as shown by Figure 2, is large enough to warrant improvement of Equation (23). Therefore the cubic term in Equation (21) is considered in Appendix D in deriving the closer approximation:

$$\hat{\Phi}(A,0) = (1.25\sqrt{A} + .24)\, e^{i \arctan\left(1 + \frac{.38}{\sqrt{A}}\right)} \tag{24}$$

APPLICATION TO DESIGN STUDIES

The major concern of a designer is the effect of aerodynamic asymmetry on dispersion. This requires a knowledge of the magnitude of $J_A \Phi$. For unit misalignment angle, ϵ, the magnitude of J_A can be estimated from Equation (12). The roll damping coefficient, C, can usually be estimated to reasonable accuracy. The problem reduc then to calculating an upper bound on $|\Phi|$ for expected values of initial and steady state spin. For those missiles whose spin magnitude is at least one rotation per distance to target and whose spin direction does not reverse in flight, expressions for this upper bound can be obtained from Equations (19) and (24). This pair of equations yields the following convenient rule-of-thumb, applicable for values of $\phi_\infty' C^{-1}$ greater than 2 radians:

Rule: The magnitude of Φ is equal to or less than the smaller of the two expressions

1) $\frac{1}{\phi_0'}$

2) $\frac{1.25}{\sqrt{\phi_\infty' C}} + \frac{.24}{\phi_\infty'}$

where ϕ_0', ϕ_∞' and C are measured per unit distance and angles are measured in terms of radians.

SUMMARY

The jump due to aerodynamic asymmetry has been studied in some detail. Curves for its magnitude and orientation which should prove useful to a designer have been computed.

Charles H. Murphy
CHARLES H. MURPHY

James W. Bradley
JAMES W. BRADLEY

APPENDIX A

DERIVATION OF EQUATION (14)

From Equations (3) and (13) we can write the jump function in the form:

$$\Phi = \lim_{\hat{s}\to\infty} C^{-1} \int_0^{\hat{s}} \left(1 - \frac{\hat{s}_1}{\hat{s}}\right) e^{i\hat{\phi}(\hat{s}_1)} \, d\hat{s}_1 \tag{A1}$$

$$\hat{\phi}(\hat{s}) = \phi(C^{-1}\hat{s}) = A\left[\hat{s}_1 - b\left(e^{-\hat{s}_1} - 1\right)\right] \tag{A2}$$

where $\hat{s} = Cs$

$b = B-1$

The integral in Equation (A1) can be simplified by replacing the real variable $\hat{s}_1$ by the complex variable $z = x + iy$ and considering the integral over the following contour:

1. Along the real axis from 0 to $\hat{s}$
2. On the circular arc $z = \hat{s}e^{i\theta}$ for θ varying from 0 to $\frac{\pi}{2}$
3. Along the imaginary axis from $\hat{s}$ to 0.

Since the integrand is analytic within this contour, the integral vanishes.

$$\therefore \oint \left(1 - \frac{z}{\hat{s}}\right) e^{i\hat{\phi}(z)} \, dz$$

$$= \int_0^{\hat{s}} \left(1 - \frac{x}{\hat{s}}\right) e^{i\hat{\phi}(x)} \, dx$$

$$+ \hat{s}\, i \int_0^{\frac{\pi}{2}} \left(1 - e^{i\theta}\right) e^{i\hat{\phi}(\hat{s}e^{i\theta})} e^{i\theta} \, d\theta$$

$$+ i \int_{\hat{s}}^{0} \left(1 - \frac{iy}{\hat{s}}\right) e^{i\hat{\phi}(iy)} \, dy$$

$$= 0 \tag{A3}$$

In order to deal with these integrals we will make use of the following relations:

$$\left|1 - e^{i\theta}\right| = \sqrt{2(1 - \cos\theta)}$$

$$\leq \sqrt{2(\theta^2/2)} = \theta \qquad (A4)$$

$$\left|e^{w}\right| = e^{R\{w\}} \qquad (A5)$$

Where $R\{w\} = u$ when $w = u + iv$

$$R\left\{i\hat{\beta}(\hat{s}e^{i\theta})\right\} = AR\left\{i\hat{s}e^{i\theta} - ib(e^{-\hat{s}e^{i\theta}} - 1)\right\}$$

$$= -A\hat{s}\sin\theta - AbR\left\{ie^{-\hat{s}e^{i\theta}}\right\}$$

$$\leq -A\hat{s}\sin\theta + A|b| \qquad (A6)$$

When $0 \leq \theta \leq \frac{\pi}{2}$

$$-\sin\theta \leq -\frac{2\theta}{\pi} \qquad (A7)$$

$$R\left\{i\hat{\beta}(iy)\right\} = AR\left\{-y - ib(e^{-iy} - 1)\right\}$$

$$\leq -Ay + A|b| \qquad (A8)$$

With these relations, upper bounds for the magnitudes of the second integral and part of the third integral in equation (A3) can be computed.

$$\left|\hat{s}i\int_0^{\frac{\pi}{2}} (1 - e^{i\theta})\, e^{i\hat{\beta}(\hat{s}e^{i\theta})}\, e^{i\theta}\, d\theta\right| \leq \hat{s}\int_0^{\frac{\pi}{2}} \left|1 - e^{i\theta}\right| e^{R\{i\hat{\beta}\}} d\theta$$

$$\leq \hat{s}\int_0^{\frac{\pi}{2}} \theta\, e^{\left(-\frac{2A\hat{s}\theta}{\pi} + A|b|\right)} d\theta$$

$$= \frac{\pi^2 e^{A|b|}}{(2A)^2\,\hat{s}}\left[-e^{-A\hat{s}}(A\hat{s} + 1) + 1\right] \qquad (A9)$$

$$\left| \int_{\hat{s}}^{0} \frac{iy}{\hat{s}} e^{i\hat{\phi}(iy)} \, dy \right| \leq \frac{1}{\hat{s}} \int_{0}^{\hat{s}} y e^{(-Ay + A|b|)} \, dy$$

$$= \frac{e^{A|b|}}{A^2 \hat{s}} \left[- e^{-A\hat{s}} (A\hat{s} + 1) + 1 \right] \tag{A10}$$

It should be noted that as $\hat{s}$ increases, both of the above integrals approach zero. Thus if the limit of the integrals in Equation (A3) for $\hat{s} \to \infty$ is taken, and y is replaced by r, the following equation can be written:

$$\Phi = \frac{iA}{\phi_\infty'} \int_{0}^{\infty} e^{Af(r)} \, dr \tag{A11}$$

$$\text{where } f(r) = \frac{i}{A} \phi \, (iC^{-1} r)$$

$$= - \left[r + ib(e^{-ir} - 1) \right]$$

$$= - \left[r + i(B-1)(e^{-ir} - 1) \right]$$

APPENDIX B

POWER SERIES EXPANSION OF $\hat{\Phi}$

For small values of A, a power series expansion can be used.

$$\hat{\Phi} = iA \int_0^\infty e^{Af(r)}\, dr$$

$$= iAe^{iAb} \int_0^\infty e^{-Ar} \sum_{n=0}^{\infty} \frac{(-iAbe^{-ir})^n}{n!}\, dr$$

$$= iAe^{iAb} \sum_{n=0}^{\infty} \frac{-(-iAb)^n e^{-r(A+ni)}}{n!\,(A+ni)} \Bigg|_0^\infty$$

$$= iAe^{iAb} \sum_{n=0}^{\infty} \frac{(-iAb)^n}{n!\,(A+ni)}$$

$$= iAe^{iAb} \left[\frac{1}{A} - \frac{iAb}{A+i} - \frac{(Ab)^2}{2(A+2i)} + \ldots\right]$$

$$= e^{iAb} \left[i + \frac{A^2 b}{A+i} - \frac{iA^3 b^2}{2(A+2i)} - \ldots\right] \qquad \text{(B1)}$$

APPENDIX C

ASYMPTOTIC SERIES EXPANSION OF $\hat{\Phi}$

In order to obtain an asymptotic series for $\hat{\Phi}$, we make repeated use of integration by parts.

$$\hat{\Phi}(A,B) = iA \int_0^\infty e^{Af(r)}\, dr$$

$$= i \int_0^\infty \frac{e^{Af}}{f'} Af'\, dr$$

$$= -\frac{i}{f'(0)} + i \int_0^\infty \left(-\frac{1}{f'}\right)' \frac{e^{Af} Af' dr}{Af'}$$

$$= i \sum_{k=0}^{n} A^{-k} F_k(0) + iR_n \qquad \text{(C1)}$$

where $F_0 = -\frac{1}{f'}$

$$F_k = F_0 F'_{k-1}$$

$$R_n = A^{-n} \int_0^\infty F'_n e^{Af}\, dr$$

By direct substitution we can get the expansion for $n = 2$.

$$\hat{\Phi} = i\left\{\frac{1}{B} + \frac{i(B-1)}{AB^3} - \frac{(B-1)(2B-3)}{A^2B^5} + \frac{1}{A^2}\int_0^\infty F'_2\, e^{Af}\, dr\right\} \qquad \text{(C2)}$$

$$\therefore \lim_{A\to\infty} \hat{\Phi} = \frac{e^{i(90^\circ)}}{B} \qquad \text{(C3)}$$

Note that the other terms in Equation (C2) grow as $B \to 0$ and hence convergence for $A \to \infty$ will be slower for smaller values of B.

APPENDIX D

DERIVATION OF EQUATION (24)

If we consider the cubic term in Equation (21), Equation (16) can be written as

$$\hat{\Phi}(A,0) = iA\int_0^\infty e^{-\frac{iAr^2}{2}-\frac{Ar^3}{6}}\,dr$$

$$\doteq iA\int_0^\infty e^{-\frac{iAr^2}{2}}\left(1-\frac{Ar^3}{6}\right)dr$$

$$= iA\int_0^\infty e^{-\frac{iAr^2}{2}}\,dr + \frac{1}{3}\int_0^\infty \left(\frac{iAr^2}{2}\right) e^{-\frac{iAr^2}{2}}\,(iArdr)$$

$$= \frac{1}{2}\sqrt{\pi A}\,(1+i) + \frac{1}{3}$$

$$= \frac{1}{2}\sqrt{\pi A} + i\left(\frac{1}{2}\sqrt{\pi A} + \frac{1}{3}\right) \qquad \text{(D1)}$$

Thus, if we express $\hat{\Phi}$ in the polar form

$$\hat{\Phi} = |\hat{\Phi}|\,e^{i\theta} \qquad \text{(D2)}$$

then
$$\left|\hat{\Phi}(A,0)\right| = \left[\frac{\pi A}{4} + \left(\frac{\pi A}{4} + \frac{\sqrt{\pi A}}{3} + \frac{1}{9}\right)\right]^{1/2}$$

$$= \left[\frac{(3\sqrt{\pi A}+1)^2+1}{18}\right]^{1/2}$$

$$\doteq \frac{3\sqrt{\pi A}+1}{\sqrt{18}}$$

$$= \sqrt{\frac{\pi A}{2}} + \frac{\sqrt{2}}{6}$$

$$\doteq 1.25\sqrt{A} + .24 \qquad \text{(D4)}$$

$$\text{and } \tan \Theta = \frac{\frac{1}{2}\sqrt{\pi A} + \frac{1}{3}}{\frac{1}{2}\sqrt{\pi A}}$$

$$= 1 + \frac{2}{3\sqrt{\pi A}}$$

$$= 1 + \frac{.38}{\sqrt{A}} \qquad \text{(D5)}$$

The following table reveals the improvement of these results over the simpler approximation, Equation (23).

	Magnitude of $\hat{\Phi}$			Orientation of $\hat{\Phi}$ (DEG)		
A	Eq.(23)	Eq.(D4)	ORDVAC	Eq.(23)	Eq.(D5)	ORDVAC
1	1.25	1.49	1.51	45	54.0	59.2
3	2.17	2.41	2.42	45	50.6	52.2
5	2.80	3.04	3.05	45	49.4	50.4
7	3.32	3.55	3.56	45	48.8	49.5

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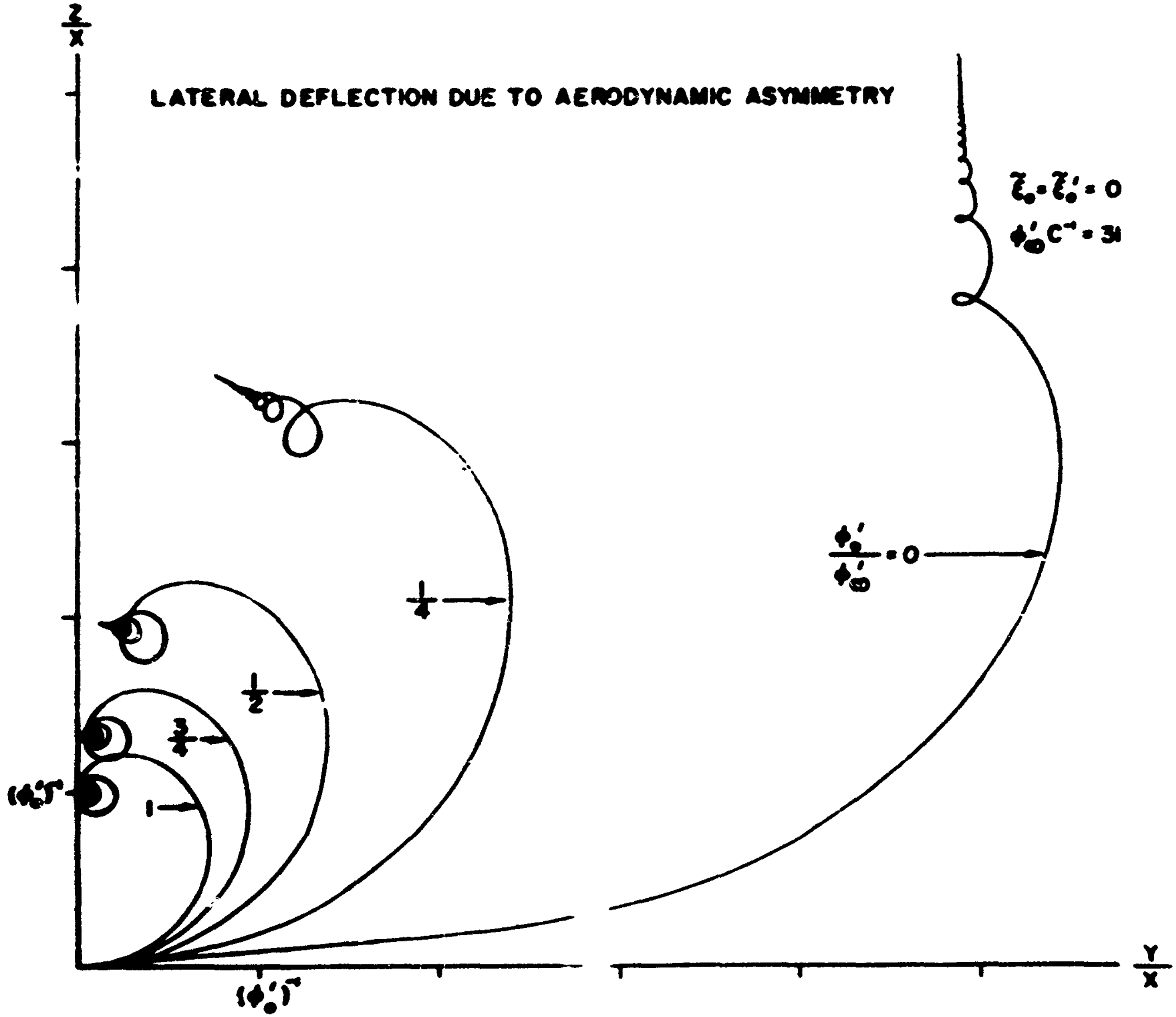
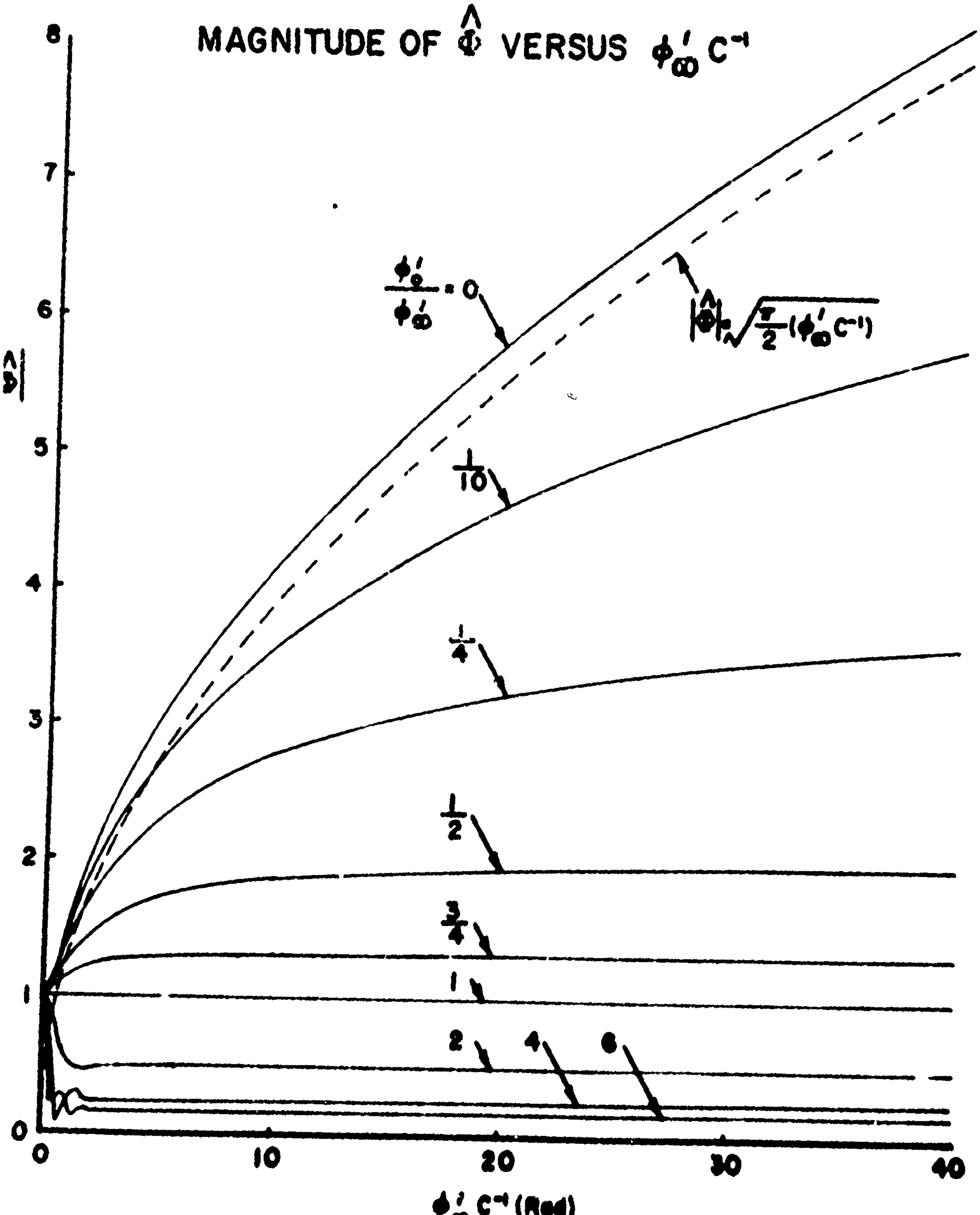


LATERAL DEFLECTION DUE TO AERODYNAMIC ASYMMETRY
$\frac{Z}{X}$
$\frac{Y}{X}$
$\zeta_0 = \zeta_0' = 0$
$\phi_\infty' C^{-1} = 31$
$\frac{\phi_0'}{\phi_\infty'} = 0$
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{3}{4}$
1
$(\phi_0')^{-1}$
$(\phi_0')^{-1}$

FIG. 1



MAGNITUDE OF $\hat{\Phi}$ VERSUS $\phi'_{\infty} C^{-1}$
$\frac{\phi'_o}{\phi'_{\infty}} = 0$
$\frac{1}{10}$
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{3}{4}$
1
2
4
6
$|\hat{\Phi}| = \sqrt{\frac{\pi}{2}(\phi'_{\infty} C^{-1})}$
$|\hat{\Phi}|$
$\phi'_{\infty} C^{-1}$ (Rad)
0
10
20
30
40
1
2
3
4
5
6
7
8

FIG. 2

ORIENTATION OF $\hat{\Phi}$ VERSUS $\phi'_\infty C^{-1}$

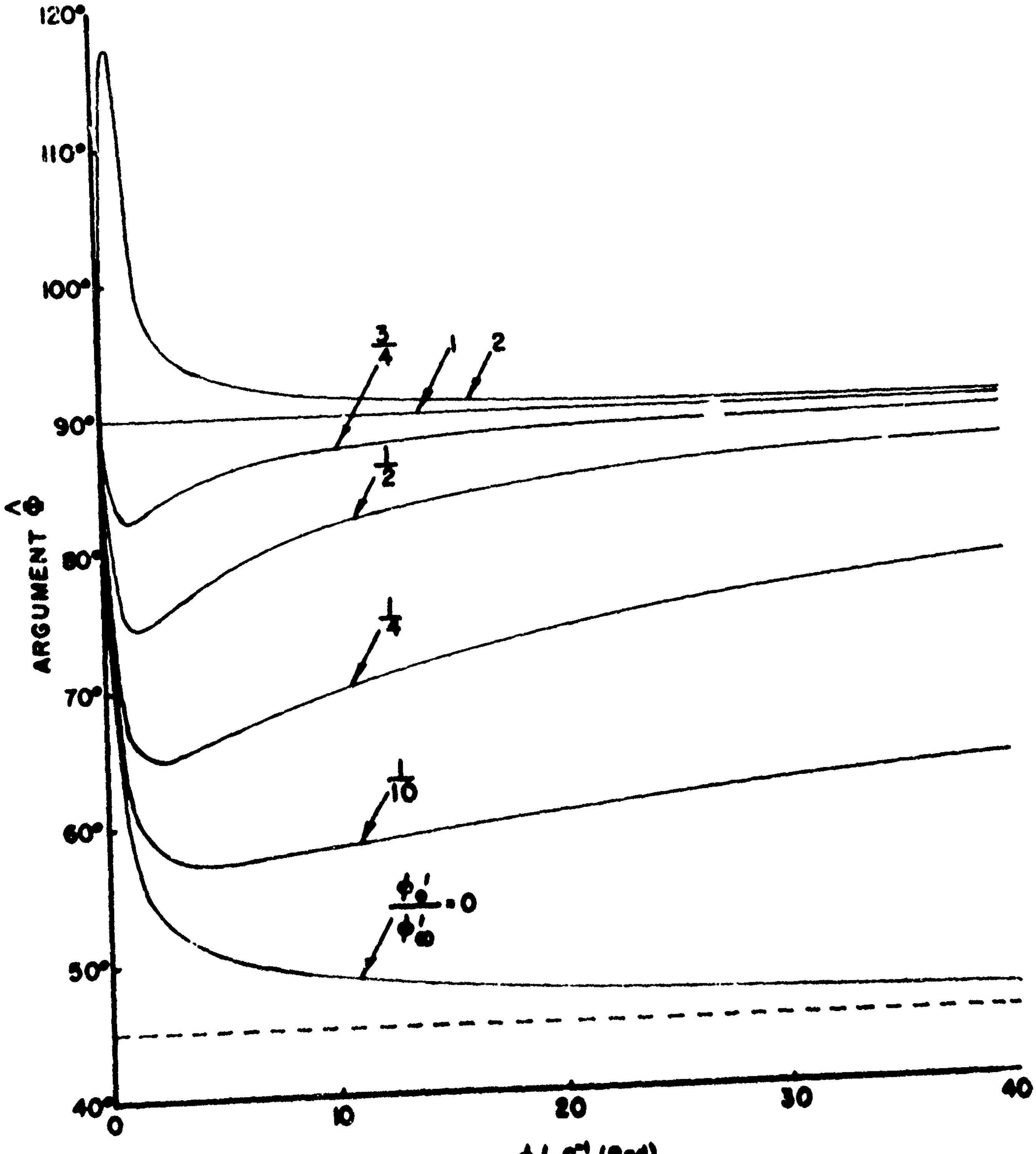
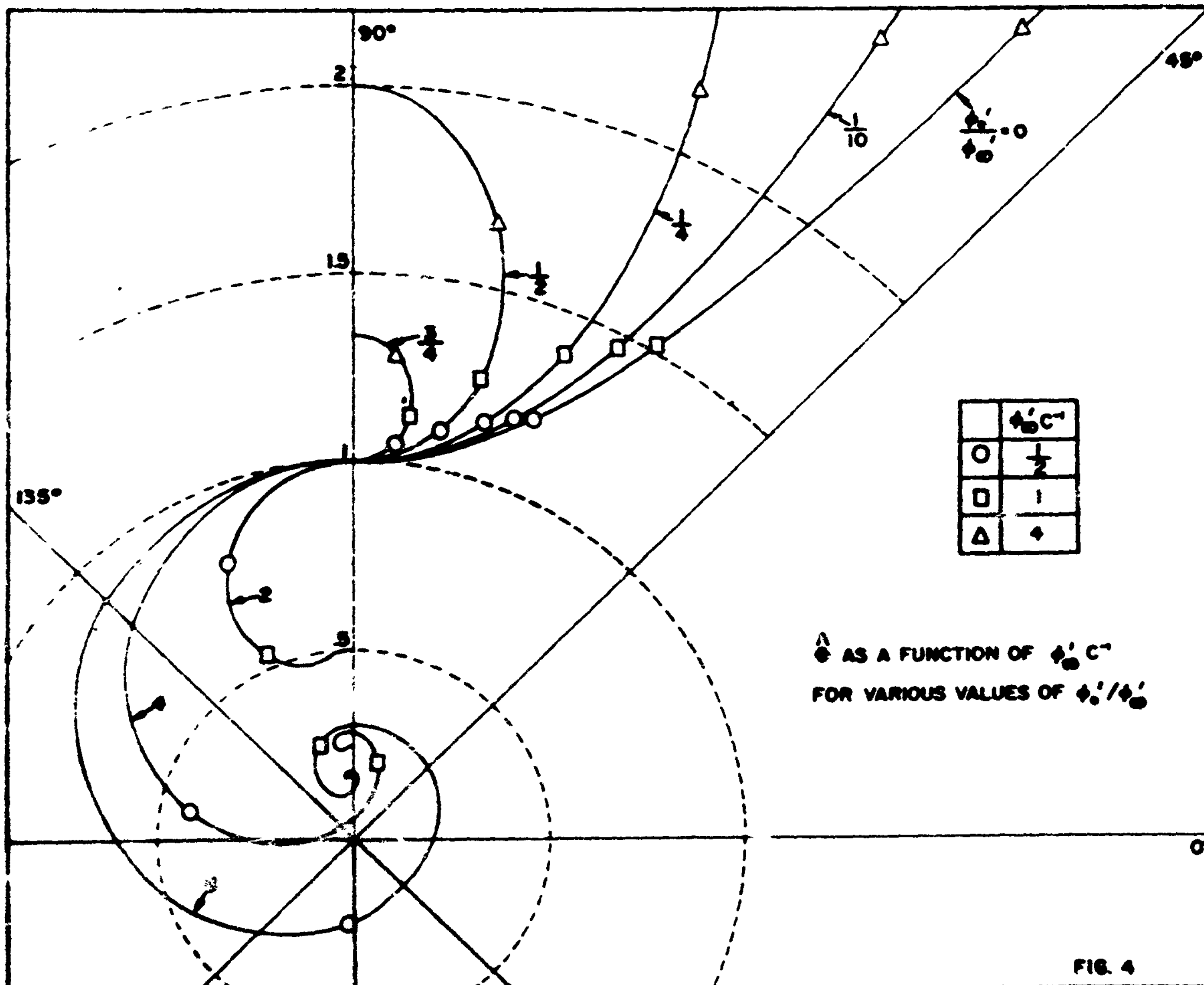


FIG. 3



$\hat{\theta}$ AS A FUNCTION OF $\phi_\infty' C^{-1}$ FOR VARIOUS VALUES OF ϕ_0'/ϕ_∞'

FIG. 4